A note on spacelike and timelike compactness

Ko Sanders*
Enrico Fermi Institute
University of Chicago
5640 South Ellis Avenue
Chicago, IL 60637

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Abstract

When studying the causal propagation of a field ϕ in a globally hyperbolic spacetime M, one often wants to express the physical intuition that ϕ has compact support in spacelike directions, or that its support is a spacelike compact set. We compare a number of logically distinct formulations of this idea, and of the complementary idea of timelike compactness, and we clarify their interrelations. E.g., a closed set $A \subset M$ has a compact intersection with all Cauchy surfaces if and only if $A \subset J(K)$ for some compact set K. (However, it does not suffice to consider only those Cauchy surfaces that partake in a given foliation of M.) Similarly, a closed set $A \subset M$ is contained in a region of the form $J^+(\Sigma^-) \cap J^-(\Sigma^+)$ for two Cauchy surfaces $\Sigma^\pm$ if and only if the intersection of A with $J(K)$ is compact for all compact K. We also treat future and past compact sets in a similar way.

1 Introduction

Suppose ϕ is a physical field configuration on a globally hyperbolic spacetime M, i.e. it is a (possibly distributional) section of some vector bundle V over M. When ϕ satisfies a normally hyperbolic equation of motion with compactly supported initial data, then the support of ϕ is contained in $J(K)$ for some compact $K \subset M$ and hence it has a compact intersection with every Cauchy surfaces. Such solutions occur often in the physics literature and are sometimes described as being "compactly supported on all Cauchy surfaces". However, when ϕ is subject to a gauge symmetry, the properties of ϕ are usually not uniquely determined by its initial data, because one may always add gauge terms with largely uncontrolled behaviour in the future or past. In this case it is less obvious whether the criterion of compact support on all Cauchy surfaces still correctly encodes the physical intuition that ϕ is "spacelike compactly supported". This problem was encountered explicitly by [5] in the context of linearised general relativity. There the authors opted for the apparently stronger criterion that ϕ has support in $J(K)$ for some compact $K \subset M$.

In this note we will consider several distinct formulations of the idea that ϕ has a spacelike compact support and we clarify their interrelations. In particular we show the equivalence of the two formulations above (after making them more precise). Furthermore, treating ϕ as a distribution (density) and assuming it has a spacelike compact support, the natural class of smooth testing sections of V consists of the ones which have timelike compact support. This leads us to consider also several distinct notions of timelike compactness, in order to clarify their relations. In addition we will take the time orientation of M into account and treat future, resp. past, compact supports along similar lines.

First, we consider a purely geometric situation, focussing on closed subsets of M. In Sec. 2, we discuss spacelike compact sets, together with future and past spacelike compact sets. Sec. 3

*E-mail: kosanders@uchicago.edu

deals with timelike compact sets, together with future or past compact sets. After these geometric preliminaries we consider in Sec. 4 conditions on distribution densities ϕ and on test-sections f, that guarantee that their supports are spacelike compact. We also introduce natural topologies on the spaces of future, past spacelike and timelike compactly supported sections and distribution densities, so that they become each others topological duals. We conclude our note in Section 5 with the special case where ϕ solves a normally hyperbolic equation and we comment on the continuity of the unique advanced and retarded fundamental solutions of such an operator w.r.t. the topologies on sections and distributions with suitable supports.

Throughout we will use standard notions and notions from Lorentzian geometry (e.g. [8]). Recall in particular that a Cauchy surface $\Sigma \subset M$ is a subset which is intersected exactly once by every inextendible timelike curve. We will assume that M is globally hyperbolic, which means that it has a Cauchy surface [2]. In addition we assume that a time-orientation for M has been fixed. As a matter of notation, we will let $\mathfrak{C}(M)$ denote the set of all Cauchy surfaces in M and $\mathfrak{C}_0(M)$ is the subset of all spacelike Cauchy surfaces. The space of smooth sections of the vector bundle V over M will be denoted by $\Gamma(M, V)$, while $\Gamma_0(M, V)$ denotes the space of compactly supported smooth sections, both in their usual topologies (cf. [1]). We let $\mathcal{D}(M, V^*)$ denote the space of distribution densities with values in the dual vector bundle V^* of V (so that on an oriented spacetime M, $\Gamma(M, V^*) \subset \mathcal{D}(M, V^*)$ by the natural pairing $\langle \phi, f\rangle := \int_M \phi(f) d\mathrm{vol}_g$, where $d\mathrm{vol}_g$ is the volume form induced by the metric g).

2 Spacelike compact sets

In this section we prove our main geometric result on spacelike compact sets and its corollary on future and past spacelike compact sets. The technical heart of these results is contained in the following proposition:

Proposition 2.1 *Let $A \subset M$ be a closed set such that $A \cap \Sigma$ is compact (in Σ, or, equivalently, in M) for all $\Sigma \in \mathfrak{C}_0(M)$. Then there is a compact set $K \subset M$ such that $A \subset J(K)$.*

A proof of this proposition is given at the end of this section. First, however, we will discuss its consequences for spacelike compactness.

Theorem 2.2 (Spacelike compact sets) *For a closed set $A \subset M$ in a globally hyperbolic spacetime the following conditions are equivalent:*

1. *There is a compact set $K \subset M$ such that $A \subset J(K)$.*
2. *For every $\Sigma \in \mathfrak{C}(M)$, $A \cap \Sigma$ is compact.*
3. *For every $\Sigma \in \mathfrak{C}_0(M)$, $A \cap \Sigma$ is compact.*

Note in particular that this dispels the concern of [5] Footnote 'b', that the first two items might not be equivalent.

Proof: It is a well-known result in Lorentzian geometry that the first condition implies the second ([1] Corollary A.5.4). The second implies the third trivially and the third implies the first by Proposition 2.1. □

These results motivate the following definition:

Definition 2.3 *We call a subset $A \subset M$* spacelike compact *when $\overline{A}$ satisfies any of the equivalent conditions of Theorem 2.2.*

In Theorem 2.2 it does not suffice to consider only the Cauchy surfaces of a given foliation of M. The following is an easy counterexample:[1]

[1] [6], Footnote 17, already gives a counterexample consisting of a set $B \subset M$ which has compact intersection with all Cauchy surfaces of a given foliation, but which is not spacelike compact. However, that set B is not closed and $\overline{B}$ seems too pathological to occur as the support of a smooth section.

Example 2.4 *Consider the Minkowski spacetime M_0 in standard inertial coordinates $(t, \mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^{d-1}$ for some $d \geq 2$. We use the foliation of M_0 by the constant t Cauchy surfaces Σ_t. For the set A we choose the support of the function $\phi(t, \mathbf{x}) := \psi(3e^{\|\mathbf{x}\|^2} t - 3)$, where $\psi \in C_0^\infty(\mathbb{R})$ has support $[-1, 1]$. This means that*

$$A = \mathrm{supp}(\phi) = \left\{ (\mathrm{t}, \mathbf{x}) | \ \frac{2}{3} \leq \mathrm{e}^{\|\mathbf{x}\|^2} \mathrm{t} \leq \frac{4}{3} \right\}.$$

It is easy to see that $A \cap \Sigma_t$ is compact for all $t \in \mathbb{R}$. Now consider the hypersurface $\Sigma := \left\{ (e^{-\|\mathbf{x}\|^2}, \mathbf{x}) \right\}$. One may show that Σ is a spacelike Cauchy surface (cf. [2] Corollary 11). To conclude the counterexample we note that $\Sigma \subset A$, so $A \cap \Sigma = \Sigma$, which is not compact. Hence, A is not spacelike compact. $\oslash$

Taking the time-orientation of M into account we define the following refined notions of spacelike compactness:

Definition 2.5 *We call a subset $A \subset M$* future*, resp.* past*,* spacelike compact *when $\overline{A} \subset J^-(K)$, resp. $\overline{A} \subset J^+(K)$, for some compact $K \subset M$.*

Note that, informally speaking, the adjectives future, past and spacelike refer to the regions of spacetime which do not intersect A. Future and past spacelike compact sets are spacelike compact. A closed set is both future and past spacelike compact if and only if it is compact.

Corollary 2.6 *For a closed set $A \subset M$ the following conditions are equivalent:*

1. *A is future (resp. past) spacelike compact.*
2. *$A \cap J^+(\Sigma)$ (resp. $A \cap J^-(\Sigma)$) is compact for every $\Sigma \in \mathfrak{C}(M)$.*
3. *$A \cap J^+(\Sigma)$ (resp. $A \cap J^-(\Sigma)$) is compact for every $\Sigma \in \mathfrak{C}_0(M)$.*

Proof: It is well-known that the first condition implies the second ([1] Corollary A.5.4). The second implies the third trivially. The third condition implies that $A \cap \Sigma$ is compact for every $\Sigma \in \mathfrak{C}_0(M)$, so $A \subset J(L)$ for some compact $L \subset M$, by Proposition 2.1. Furthermore, choosing a foliation of M by spacelike Cauchy surfaces Σ_t (cf. [3]) and using the fact that for any $t \in \mathbb{R}$ the set $A \cap J^+(\Sigma_t)$ (resp. $A \cap J^-(\Sigma_t)$) is compact, we may find a T such that $A \subset J^-(\Sigma_T)$ (resp. $J^+(\Sigma_T)$). Choosing $K := J(L) \cap \Sigma_T$ we find $A \subset J^-(K)$ (res. $A \subset J^+(K)$), proving the future (resp. past) spacelike compactness. $\square$

To conclude this section we supply the proof of Proposition 2.1. We begin with a lemma, which uses an exhaustion by compact sets ([7] Proposition 4.76):

Lemma 2.7 *Let $\Sigma \in \mathfrak{C}_0(M)$ and let $\{K_n\}_{n \in \mathbb{N}}$ be an exhaustion of Σ by compact sets, i.e. each $K_n \subset \Sigma$ is compact, $K_n \subset \mathring{K}_{n+1}$ and $\cup_{n \in \mathbb{N}} K_n = \Sigma$. Assume that there are sequences of points $x_n \in M$ and compact spacelike acausal submanifolds $B_n \subset M$ with boundary, such that*

1. *$x_n \in B_n$,*
2. *$J(B_n) \cap \Sigma \subset \mathring{K}_n$,*
3. *$J(B_{n+1}) \cap K_n = \emptyset$.*

Then there is a $\Sigma' \in \mathfrak{C}_0(M)$ which contains all B_n, and the set $X := \bigcup_{n \in \mathbb{N}} \{x_n\}$ is closed, but not compact.

Proof: We may construct a spacelike Cauchy surface $\Sigma' \subset M$ that contains all B_n as follows. First we define $L_1 := K_1$ and by induction we choose compact subsets $L_n \subset K_n$, $n \geq 2$, such that $J(B_n) \cap \Sigma \subset \mathring{L}_n$, but $L_n \cap K_{n-1} = \emptyset$. (This is possible, by our assumptions on B_n and K_n.)

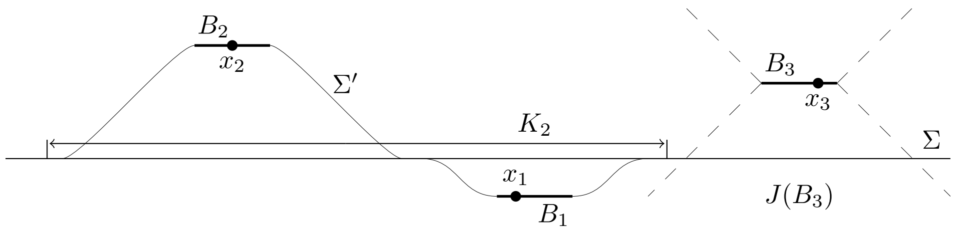

Figure 1: A schematic depiction of the geometric construction used to prove Lemma 2.7.

Note in particular that all L_n are pairwise disjoint. The idea is that the domain of dependence $M_n := D(\mathring{L}_n)$ provides some room around B_n to deform the Cauchy surface Σ, whilst the K_n ensure that the B_n do not accumulate (see Figure 2).

For each $n \in \mathbb{N}$ the region M_n is a globally hyperbolic spacetime in its own right ([1] Lemma A.5.9). We may choose spacelike Cauchy surfaces S_n for M_n such that $B_n \subset S_n$, by [3] Theorem 1.1. We then set

$$\Sigma' := \left(\Sigma \setminus \bigcup_{n \in \mathbb{N}} \mathring{L}_n \right) \cup \bigcup_{n \in \mathbb{N}} S_n.$$

To prove that Σ' is a Cauchy surface for M we let γ be an arbitrary inextendible timelike curve. If γ does not intersect Σ in some $\mathring{L}_n$, then it already intersects Σ'. Moreover, this point of intersection is unique, as γ cannot intersect any of the M_n. On the other hand, if γ intersects some $\mathring{L}_n$, then it cannot intersect Σ' in $\Sigma \cap \Sigma'$ or in any S_k with $k \neq n$. Furthermore, γ intersects M_n and the intersection is an inextendible causal curve in M_n, which has a unique point of intersection with S_n. Therefore, Σ' is a Cauchy surface. Also note that Σ' contains all B_n, by construction, and that it is spacelike, because Σ is spacelike.

To conclude the proof we show that $X \subset \Sigma'$ is closed but not compact. First suppose that $y \in \overline{X}$ and let $U \subset \Sigma'$ be a compact neighbourhood of y. Note that $J(U) \cap \Sigma \subset K_N$ for some $N \in \mathbb{N}$. By construction, K_N does not intersect L_n with $n > N$, so $D(K_N) \cap \Sigma'$ does not contain x_n with $n > N$. It follows that y must be one of the points $x_1, \ldots, x_N$, so X is closed. Now consider the open cover of X consisting of the sets $\{S_n, n \in \mathbb{N}\}$. Each x_n is contained only in the corresponding S_n, so there is no proper subcover. This proves in particular that there is no finite subcover, so $X \subset \Sigma'$ is not compact. □

We may now prove Proposition 2.1:

Proof: We will assume that there is no set K such that $A \subset J(K)$ and derive a contradiction. For this purpose we fix a $\Sigma \in \mathfrak{C}_0(M)$ and an exhaustion of Σ by compact sets $\{K_j\}_{j \in \mathbb{N}}$. We consider the set $\dot{A} := A \setminus \Sigma$ and note that $\dot{A}$ is not contained in any set of the form $J(L)$ with compact $L \subset \Sigma$ (otherwise we could take $K = L \cup (A \cap \Sigma)$). In particular, $\dot{A} \neq \emptyset$, so we may choose $x_1 \in \dot{A}$ and $j_1 \in \mathbb{N}$ such that $x_1 \in D(\mathring{K}_{j_1})$. We now proceed by induction to choose sequences of points $x_n \in \dot{A}$ and numbers $j_n \in \mathbb{N}$ such that $x_n \in D(\mathring{K}_{j_n})$ and $J(x_{n+1}) \cap K_{j_n} = \emptyset$. This is possible, because for each n, $\dot{A} \setminus J(K_{j_n})$ contains some point x_{n+1} and the compact set $J(x_{n+1}) \cap \Sigma$ is contained in the interior of some $K_{j_{n+1}}$.

Note that $n \mapsto j_n$ is strictly increasing, so K_{j_n} is again an exhaustion of Σ by compact sets. Using this in Lemma 2.7 with $B_n := \{x_n\}$ yields a spacelike Cauchy surface Σ' containing all x_n, but for which $A \cap \Sigma' \supset X$ is not compact. This is the desired contradiction. □

3 Timelike compact sets

We now turn to the complementary notion of timelike compact sets. In this case our main geometric result is

Theorem 3.1 (Future and past compact sets) *For a closed set $A \subset M$ in a globally hyperbolic spacetime the following conditions are equivalent:*

1. *There is a Cauchy surface $\Sigma \subset M$ such that $A \subset J^+(\Sigma)$ (resp. $A \subset J^-(\Sigma)$).*

2. *For every compact set $K \subset M$, the set $A \cap J^-(K)$ (resp. $A \cap J^+(K)$) is compact.*

3. *For every point $p \in M$, the set $A \cap J^-(p)$ (resp. $A \cap J^+(p)$) is compact.*

Proof: For any compact set $K \subset M$ and any Cauchy surface $\Sigma \subset M$, the sets $J^\pm(K)$ are closed and the intersection $J^\pm(K) \cap J^\mp(\Sigma)$ is compact (cf. [1] Lemma A.5.4, Lemma A.5.1 and the comment above Lemma A.5.7). It then follows immediately that the first condition implies the second. The second implies the third trivially. It only remains to show that the third condition implies the first.

By a reversal of time-orientation it suffices to consider the case where $A \cap J^-(p)$ is compact for all $p \in M$. We choose a global time function t on M and a foliation $M \simeq \mathbb{R} \times \Sigma$ by Cauchy surfaces, so that t is the projection onto the first factor (cf. [3]). For each inextendible timelike curve γ in M we then define

$$t_-(\gamma) := \min\left\{0; t(x), x \in \gamma \cap J^+(A)\right\}$$

The minimum $t_-(\gamma)$ exists, because if $x \in \gamma \cap J^+(A)$, then $\gamma \cap J^+(A) \cap J^-(x)$ is compact and $t_-(\gamma)$ is the minimum value of t on this set.

Now consider the inextendible timelike curves $\gamma_p(t) := (t, p)$, define $T_-(p) := t_-(\gamma_p)$ and consider the embedding $\psi_- : \Sigma \to M$ by $\psi_-(p) := (T_-(p), p)$. The image Σ_- of ψ_- has the following properties. Firstly, if $(t, p) \in A$, then $T_-(p) \le t$ by construction, so $A \subset J^+(\Sigma_-)$. Secondly, Σ_- is achronal, for if there were a timelike curve γ_1 between, say, $(T_-(p), p)$ and $(T_-(q), q)$ with $T_-(q) \ge T_-(p)$, and if γ_2 is a causal curve from some point $x \in A$ to $(T_-(p), p)$, then the concatenation of γ_1 and γ_2 can be deformed to a time-like curve from x to $(T_-(q), q)$ (cf. [8]). Hence, $T_-(q)$ cannot be the minimum as defined, leading to a contradiction. (If no such γ_2 exists, then $T_-(p) = T_-(q) = 0$ and γ_1 cannot exist either.) Thus we see that Σ_- is achronal. Finally, Σ_- is a Cauchy surface. To prove this we consider an inextendible causal curve $\tau \mapsto \gamma(\tau)$ in M. There is a unique point $p \in \Sigma$ such that $(t_-(\gamma), p) \in \gamma$. Both when $\gamma \cap J^+(A) = \emptyset$ and when $\gamma \cap J^+(A) \neq \emptyset$ one may see that $(t_-(\gamma), p) \in \Sigma_-$, by an argument that involves the concatenation of causal curves as above, together with the definition of $T_-(p)$. Therefore, γ intersects Σ_-, and as Σ_- is achronal, the point of intersection is unique. This proves that Σ_- is a Cauchy surface with $A \subset J^+(\Sigma_-)$, so we established the first condition. □

Definition 3.2 *We call a subset $A \subset M$* future*, resp.* past*,* compact *when there is a Cauchy surface $\Sigma \subset M$ such that $A \subset J^-(\Sigma)$, resp. $A \subset J^+(\Sigma)$. We call A* timelike compact *when A is both future and a past compact.*

By Theorem 3.1, our definition of future and past compact sets is equivalent to the one in [1], at least for closed subsets of globally hyperbolic spacetimes. Using the same theorem it may easily be shown that a set is future, resp. past, spacelike compact if and only if it is both spacelike compact and future, resp. past, compact (cf. the proof of Corollary 2.6).

When $A \subset M$ is timelike compact and we consider a foliation of M by Cauchy surfaces Σ_t, it is not necessarily true that there are numbers $t_- < t_+$ such that $A \subset J^+(\Sigma_{t_-}) \cap J^-(\Sigma_{t_+})$. A counterexample in Minkowski spacetime can be obtained, using the notations of Example 2.4, by choosing A to be the image of Σ_0 under a non-trivial Lorentz boost. Clearly A itself is still a Cauchy surface and hence timelike compact, but it contains points with arbitrary values of t.

Note furthermore that in order to establish timelike compactness it does not suffice that A has a compact intersection with all inextendible causal curves. The following is a counterexample:

Example 3.3 *Consider the Minkowski spacetime M_0 in standard inertial coordinates $(t, \mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^{d-1}$ for some $d \ge 2$. The region $M' := I^+(0) \subset M$ is a globally hyperbolic spacetime in*

its own right and the hypersurfaces $\Sigma_R := \left\{t = \sqrt{R^2 + \|\mathbf{x}\|^2}\right\}$*,* $R > 0$*, foliate* M' *by Cauchy surfaces. Note that* M' *cannot contain a Cauchy surface for* M*, because for any* $\mathbf{x}$ *with unit norm, the inextendible timelike curve* $\gamma_{\mathbf{x}}(\tau) := (\sinh(\tau), \cosh(\tau)\mathbf{x})$ *does not enter* M'*. For the set* A *we choose the support of the function* $\phi(t, \mathbf{x}) := \psi(2\sqrt{1 + \|\mathbf{x}\|^2}(t - \sqrt{1 + \|\mathbf{x}\|^2}))$*, where* $\psi \in C_0^\infty(\mathbb{R})$ *has support* $[-1, 1]$*. Note that* A *is timelike compact in* M' *(using the foliation). However, it cannot be timelike compact in* M*, because the inextendible timelike curve* $\gamma_{\mathbf{x}}$ *lies entirely in* $J^-(A) \setminus A$*. Hence, if* $A \subset J^-(\Sigma)$ *for some Cauchy surface* Σ *and if* $x \in \gamma_{\mathbf{x}} \cap \Sigma$*, we could construct a timelike curve from* x *via* A *to* Σ*, contradicting the fact that* Σ *is Cauchy. Nevertheless, any inextendible causal curve* γ *has a compact intersection with* A*, because if* γ *does not enter* M' *the intersection is empty, while if* γ *does enter* M'*, the intersection is compact, since* A *is timelike compact in* M'*.*
$\oslash$

4 Spacelike and timelike compact supports

Now we return to the original motivation and consider a distribution density ϕ with values in some vector bundle V on M. We make the following obvious definition:

Definition 4.1 *A distribution density* ϕ *on* M *is said to have spacelike, timelike, future (spacelike), resp. past (spacelike) compact support if and only if* $\mathrm{supp}(\phi)$ *is spacelike, timelike, future (spacelike), resp. past (spacelike) compact.*

Again, it does not suffice to consider only a particular foliation of Cauchy surfaces to obtain spacelike compactness, nor does it suffice to assume compact intersections with all inextendible causal curves to obtain timelike compactness. Indeed, both of the counterexamples 2.4 and 3.3 are based on the supports of smooth sections ϕ. However, in the spacelike case we do have the following result:

Theorem 4.2 *Let* ϕ *be a distribution density on* M *and assume that either*

a) ϕ *is continuous, or*

b) $WF(\phi)$ *has no timelike vectors, so its restriction to all spacelike Cauchy surfaces is well-defined by microlocal arguments.*

Then the following conditions are equivalent:

1. ϕ *is spacelike compactly supported.*
2. *There is a compact set* $K \subset M$ *such that* $\mathrm{supp}(\phi|_\Sigma) \subset J(K)$ *for all* $\Sigma \in \mathfrak{C}(M)$*.*
3. *There is a compact set* $K \subset M$ *such that* $\mathrm{supp}(\phi|_\Sigma) \subset J(K)$ *for all* $\Sigma \in \mathfrak{C}_0(M)$*.*
4. $\mathrm{supp}(\phi|_\Sigma)$ *is compact for all* $\Sigma \in \mathfrak{C}(M)$*.*
5. $\mathrm{supp}(\phi|_\Sigma)$ *is compact for all* $\Sigma \in \mathfrak{C}_0(M)$*.*

Proof: The implications 2→3 and 4→5 are trivial. The implications 2→4 and 3→5 follow from the fact that $J(K) \cap \Sigma$ is compact for every compact $K \subset M$ and every Cauchy surface $\Sigma \subset M$ ([1] Lemma A.5.4). Furthermore, 1→2 follows from Theorem 2.2 and the fact that $\mathrm{supp}(\phi|_\Sigma) \subset \mathrm{supp}(\phi) \cap \Sigma$. To complete the proof it suffices to prove that 5→1. By Theorem 2.2 we only need to show that $\mathrm{supp}(\phi) \cap \Sigma$ is compact for all $\Sigma \in \mathfrak{C}_0(M)$. We will argue by contradiction, so we assume that there is a spacelike Cauchy surface $\Sigma \subset M$ such that $\mathrm{supp}(\phi) \cap \Sigma$ is not compact. We may foliate M by spacelike Cauchy surfaces Σ_t, $t \in \mathbb{R}$, such that the projection t on the first factor is a global time coordinate and $\Sigma = \Sigma_0$ (cf. [3] Theorem 1.2).

We can find an exhaustion of Σ by compact sets K_n and a sequence of points $x_n \in \Sigma$ such that $x_n \subset \mathring{K}_n$ and $x_{n+1} \notin K_n$, much in the same way is in the proof of Proposition 2.1. We now write $x_n = (0, q_n)$ and recall that $x_n \in \mathrm{supp}(\phi)$. For any open neighbourhood $U \subset \Sigma$ of

q_n and any $\epsilon > 0$ we may choose a test-function $\chi \in C_0^\infty(U)$ such that the distribution density $t \mapsto \phi(t, \chi)$ does not vanish identically on $(-\epsilon, \epsilon)$, by Schwartz' Kernels Theorem. Furthermore, by assumption a) or b) this distribution is at least continuous, so there is some $t_n \in (-\epsilon, \epsilon)$ for which $\phi(t_n, \chi) \neq 0$. This entails that $(t_n, q_n) \in \mathrm{supp}(\phi|_{\Sigma_{t_n}})$.

By induction we choose a sequence of numbers $\epsilon_n > 0$ which is sufficiently small to ensure that $J(\pm\epsilon_n, q_n) \subset \mathring{K}_n$ and $J(\pm\epsilon_{n+1}, q_{n+1}) \cap K_n = \emptyset$ for all n and both signs. Then, choosing $t_n \in (-\epsilon_n, \epsilon_n))$ as above, we may choose compact subsets $B_n \subset \Sigma_{t_n}$ such that $J(B_n) \cap \Sigma \subset \mathring{K}_n$ and $J(B_{n+1}) \cap K_n = \emptyset$. With these x_n, B_n and K_n the assumptions of Lemma 2.7 are satisfied, so there is a spacelike Cauchy surface Σ' containing all B_n and such that the set $X := \cup_{n\in\mathbb{N}} \{x_n\}$ is closed but not compact in Σ'. Since Σ' and Σ_{t_n} coincide in a neighbourhood of x_n, x_n is also in $\mathrm{supp}(\phi|_{\Sigma'})$. In other words, $\mathrm{supp}(\phi|_{\Sigma'}) \supset X$ and therefore $\mathrm{supp}(\phi|_{\Sigma'})$ is not a compact set. This contradicts the assumptions, hence ϕ must have spacelike compact support. □

For any closed set $B \subset M$ we may consider the space $\Gamma(B, V)$ of smooth sections of V on M with support in B, as a closed subspace of $\Gamma(M, V)$. In analogy to $\Gamma_0(M, V)$ we may then define the spaces of sections with spacelike, timelike and future, resp. past, (spacelike) compact supports as inductive limits (cf. [9]):

$$\begin{aligned}
\Gamma_{fsc}(M,V) &:= \bigcup_{K\subset M} \Gamma(J^-(K),V), & \Gamma_{fc}(M,V) &:= \bigcup_{\Sigma\subset M} \Gamma(J^-(\Sigma),V),\\
\Gamma_{psc}(M,V) &:= \bigcup_{K\subset M} \Gamma(J^+(K),V), & \Gamma_{pc}(M,V) &:= \bigcup_{\Sigma\subset M} \Gamma(J^+(\Sigma),V),\\
\Gamma_{sc}(M,V) &:= \bigcup_{K\subset M} \Gamma(J(K),V), & \Gamma_{tc}(M,V) &:= \bigcup_{\Sigma^\pm\subset M} \Gamma(J^+(\Sigma^-)\cap J^-(\Sigma^+),V),
\end{aligned}$$

where K is compact and $\Sigma, \Sigma^\pm$ are Cauchy surfaces. (For the spacelike compact case this agrees with Definition 3.4.6 of [1]. For smooth functions the topologies on $C^\infty_{sc}(M)$, $C^\infty_{fsc}(M)$ and $C^\infty_{psc}(M)$ coincide with those introduced by [4].) With these topologies, the following inclusions are continuous

$$\begin{aligned}
&\Gamma_0(M,V) \subset \Gamma_{fsc}(M,V) \subset \Gamma_{sc}(M,V) \subset \Gamma(M,V), \\
&\Gamma_0(M,V) \subset \Gamma_{tc}(M,V) \subset \Gamma_{fc}(M,V) \subset \Gamma(M,V),
\end{aligned} \tag{1}$$

and similarly with past (spacelike) compact instead of future (spacelike) compact supports.

In an analogous way we may introduce spaces of distribution densities with the same support properties, which will be indicated by the same subscripts, e.g.

$$\mathcal{D}_{sc}(M,V) = \bigcup_{K\subset M} \mathcal{D}(J(K),V),$$

where $\mathcal{D}(B, V)$ is the space of distribution densities with support in B, as a closed linear subspace of $\mathcal{D}(M, V)$ in the usual distributional topology.

Theorem 4.3 *Each of the spaces $\Gamma_*(M, V)$, where $*$ indicates any of the subscripts fsc, psc, sc, fc, pc, tc, is reflexive and we have*

$$\begin{aligned}
&\mathcal{D}_{fsc}(M,V^*) = \Gamma_{pc}(M,V)' \quad \mathcal{D}_{psc}(M,V^*) = \Gamma_{fc}(M,V)' \quad \mathcal{D}_{sc}(M,V^*) = \Gamma_{tc}(M,V)'\\
&\mathcal{D}_{fc}(M,V^*) = \Gamma_{psc}(M,V)' \quad \mathcal{D}_{pc}(M,V^*) = \Gamma_{fsc}(M,V)' \quad \mathcal{D}_{tc}(M,V^*) = \Gamma_{sc}(M,V)'.
\end{aligned}$$

Proof: Using the continuous embeddings in equation (1), any $\phi \in \Gamma_*(M, V)'$ is a distribution density. In the case $* = pc$, let $\Sigma \subset M$ be any Cauchy surface. The restriction map from $\Gamma(J^+(\Sigma), V)$ to $\Gamma(I^+(\Sigma), V)$ is continuous and it has a dense range, as may be shown by direct approximation, using multiplication with suitable cut-off functions. Therefore, the restriction of ϕ to $I^+(\Sigma)$ is continuous on $\Gamma(I^+(\Sigma), V)$, so it has compact support. It follows that $I^+(\Sigma)\cap\mathrm{supp}(\phi)$ is compact for any Σ and hence $I^+(\Sigma) \cap \mathrm{supp}(\phi)$ is compact too (since $J^+(\Sigma) \subset I^+(\Sigma')$ for some

Σ'). By Corollary 2.6 ϕ has future spacelike compact support. Conversely, if ϕ has future spacelike compact support, then we can find a smooth cut-off function $\chi \in C^{\infty}_{fsc}(M)$ such that $\chi \equiv 1$ on $\mathrm{supp}(\phi)$. The map $f \mapsto \chi f$ is continuous from $\Gamma_{pc}(M,V)$ to $\Gamma_0(M,V)$ and $\phi(f) = \phi(\chi f)$, so $\phi \in \Gamma_{pc}(M,V)'$.

The second item on the first line is proved by reversing the time-orientation. The third item is proved in a similar way, using Theorem 2.2 instead of Corollary 2.6. The items on the second line are also proved in a similar way, but now using Theorem 3.1. Finally we note that both $\Gamma(M,V)$ and $\Gamma_0(M,V)$ are reflexive. The reflexivity of all $\Gamma_*(M,V)$ then follows from the proofs above, if we interchange the roles of smooth sections and distribution densities. $\square$

5 Consequences for normally hyperbolic operators

To conclude this note we consider the case where ϕ satisfies a linear, normally hyperbolic field equation. In this case one expects that the spacelike compactness is preserved under the time evolution, so it would suffice to consider only one Cauchy surface. To be more precise,

Proposition 5.1 *If ϕ satisfies a normally hyperbolic equation, then the following are equivalent:*

1. *ϕ has spacelike compact support.*
2. $\mathrm{supp}(\phi|_{\Sigma})$ *is compact for all $\Sigma \in \mathfrak{C}(M)$.*
3. *There is a smooth spacelike Cauchy surface $\Sigma \in \mathfrak{C}_0(M)$ such that* $\mathrm{supp}(\phi) \cap \Sigma$ *is compact.*

Proof: We have already seen in Theorem 4.2 that the first and second items are equivalent and they both trivially imply the third. For the converse one uses the well-posedness of the Cauchy problem and the fact that compactness of $\mathrm{supp}(\phi) \cap \Sigma$ implies that both initial data on Σ have compact support. $\square$

Note that in this case it does suffice to consider the Cauchy surfaces Σ_t which belong to a given foliation of M and to require that $\mathrm{supp}(\phi) \cap \Sigma_t$ is compact. It clearly does not suffice to require that $\phi|_{\Sigma}$ has compact support for a single spacelike $\Sigma \in \mathfrak{C}(M)$, because the other initial datum may not have compact support. However, it is less clear whether it suffices to require that $\mathrm{supp}(\phi|_{\Sigma_t})$ is compact for all $t \in \mathbb{R}$ and a given foliation Σ_t of M.

Let P denote a normally hyperbolic operator in the vector bundle V over M and let $E^{\pm}$ denote the unique advanced and retarded fundamental operators. It is well-known [1] that these are continuous linear maps

$$E^{\pm} : \Gamma_0(M,V) \to \Gamma_{sc}(M,V)$$

such that $\mathrm{supp}(E^{\pm}f) \subset J^{\pm}(\mathrm{supp}(f))$. Using the topologies introduced in Section 4 and the support properties it is in fact not hard to show that the maps

$$\begin{aligned} E^{+} &: \Gamma_{psc}(M,V) \to \Gamma_{psc}(M,V) & E^{-} &: \Gamma_{fsc}(M,V) \to \Gamma_{fsc}(M,V) \\ E^{+} &: \Gamma_{pc}(M,V) \to \Gamma_{pc}(M,V) & E^{-} &: \Gamma_{fc}(M,V) \to \Gamma_{fc}(M,V) \end{aligned}$$

are continuous. (The proof is analogous to that of [4] Lemma 3.11). This entails e.g. that $E^{+} : \Gamma_0(M,V) \to \Gamma_{psc}(M,V)$ and $E^{+} : \Gamma_{tc}(M,V) \to \Gamma_{pc}(M,V)$ are also continuous, by the continuous inclusions (1). When M is oriented one may define the operators E^{+} also on distributional sections, by duality. We then have $(E^{\pm}\phi, f) = (\phi, E^{\mp}f)$, which leads to continuous linear maps

$$\begin{aligned} E^{+} &: \mathcal{D}_{psc}(M,V) \to \mathcal{D}_{psc}(M,V) & E^{-} &: \mathcal{D}_{fsc}(M,V) \to \mathcal{D}_{fsc}(M,V) \\ E^{+} &: \mathcal{D}_{pc}(M,V) \to \mathcal{D}_{pc}(M,V) & E^{-} &: \mathcal{D}_{fc}(M,V) \to \mathcal{D}_{fc}(M,V). \end{aligned}$$

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